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An Application of a Fixed Point Theorem to Best Approximation

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The object of this paper is twofold. First, certain fixed point theorems of Dotson [1] are generalized (Theorems 1, 2) by an appeal to an earlier result of this author [5]. Second, one of these theorems is utilized to obtain a theorem (Theorem 3) on best approximations which incidentally generalizes a result of Meinardus ([3], Theorem). This theorem has interesting applications brought out in its corollaries, and its proof reveals how the original proof of Meinardus can be considerably simplified with no appeal to algebraic-topological ideas implicit in Schauder's fixed point theorem.

1. PRELIMINARIES

If E is a linear space, as is well known, a subset S of E is said to be star-shaped with reference to $p \in S$ if for each $s \in S$, the line segment $[p, s] \subset S$. $S \subset E$ is said to be star-shaped if it is star-shaped with reference to one of its elements. A convex set is obviously star-shaped.

If E is a normed linear space, $T : S \to S$ is nonexpansive if for any pair $x, y \in S$, $\| T(x) - T(y)\| \leqslant \| x - y \|$, where $\| \cdot \|$ is the norm on E. Contraction mappings are thus nonexpansive and any nonexpansive map is continuous.

DEFINITION 1. A subset S of a normed linear space E is said to be ϵ-chainable (for a positive real number ϵ) if for any pair x, $y \in S$, we can find a finite number of elements z_i, $i = 0, 1, 2,..., n$ with $z_0 = x$ and $z_n = y$ such that $\| z_i - z_{i+1} \| \leqslant \epsilon$ for $i = 0, 1,..., n - 1$. (Thus star-shaped subsets are ϵ-chainable for every positive real number ϵ.)

DEFINITION 2 [2]. An operator T on a subset S of E, a normed linear space, mapping S into E is called an ϵ-local contraction (for a positive real number ϵ) if for each $x \in S$, we can find a positive number $\lambda(x)$ less than 1 such that for $y_1, y_2 \in \{z : \| x - z \| \leqslant \epsilon\}$, $\| T(y_1) - T(y_2)\| \leqslant \lambda \| y_1 - y_2 \|$. If λ

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does not depend on $x \in S$, T is said to be an $(\epsilon\text{-}\lambda)$ uniform local contraction operator.

Just by a clever use of the contraction mapping principle, Dotson ([1], Theorem 1) proved

THEOREM A. *If S is a compact and star-shaped subset of the normed linear space E and $T : S \to S$ is nonexpansive, then T has a fixed point.*

The import of Theorem A is that in Schauder's theorem, to the effect that any continuous map $T : S \to S$, where S is compact and convex in E, has a fixed point, we can relax the convexity assumption by requiring T to be more than continuous, viz., nonexpansive.

It may be recalled ([5], Definition 4) that $T : S \to S$ is said to be a Banach operator of type k on S if there exists a constant k, $0 \leqslant k < 1$ such that

$$\| T(x) - T^2(x) \| \leqslant k \| x - T(x) \| .$$

From [5], Corollary 2 we have

THEOREM B. *A continuous Banach operator mapping a closed subset of a Banach space into itself has a fixed point.*

From [2], Theorem 1 we have

THEOREM C. *Any $(\epsilon\text{-}\lambda)$ uniform local contraction T mapping a closed ϵ-chainable subset of a Banach space into itself has a fixed point.*

The following definition is due to Opial [4].

DEFINITION 3. A map $T : S \to E$, $S \subset E$, is said to be demiclosed if for any sequence $\{x_n\}$ in S converging weakly to x with $\{T(x_n)\}$ converging strongly to $y \in E$, $T(x) = y$.

2. GENERALIZATIONS OF DOTSON'S RESULTS

Theorems 1 and 2 below generalize Theorems 1, 2 of [1].

THEOREM 1. *Let T be a continuous operator mapping a compact subset S (of a normed linear space E) into itself. Suppose*

(i) *there exist $p \in S$ and a fixed sequence of positive numbers $\{k_n\}$ ($k_n < 1$) converging to 1, such that $(1 - k_n) p + k_n T(x) \in S$ for each $x \in S$; further for each $x \in S$ and k_n, $\| T((1 - k_n) p + k_n T(x)) - T(x) \| \leqslant \| (1 - k_n) p + k_n T(x) - x \|$; or*

(ii) *S is star-shaped with reference to $p \in S$; further* $\| T(x) - T(y) \| \leq \| x - y \|$, *whenever* $\| x - y \| \leq \epsilon$ $(x, y \in S)$ *for a positive number* ϵ.

Then T has a fixed point.

Proof. If (i) is true, then each map T_n defined by $T_n(x) = (1 - k_n) p + k_n T(x)$ is a continuous Banach operator of type k_n. Further each T_n maps S into itself in view of hypothesis (i) and so has a fixed point p_n, by Theorem B. As S is compact, $\{p_n\}$ has a subsequence $\{p_{n_k}\}$ converging to q, say. So $p_{n_k} = T_{n_k}(p_{n_k}) = (1 - k_{n_k}) p + k_{n_k} T(p_{n_k})$. Making k tend to ∞, by the continuity of T, one obtains $q = 0 \cdot p + T(q)$, since $\{k_{n_k}\} \to 1$.

If (ii) holds, then each map T_n defined above is an $(\epsilon\text{-}k_n)$ uniform local contraction mapping S into itself. Further, in view of hypothesis (ii), S is ϵ-chainable for every positive real number ϵ. So, by Theorem C, each T_n has a fixed point. The rest of the proof is as in the preceding paragraph.

Remarks. If S is star-shaped about p, then Theorem A readily follows from Theorem 1. Hypothesis (i) above is an attempt to weaken the convexity assumption of Theorem A, as illustrated in the following example.

EXAMPLE 1. Let S be the set

$$\{(0, y) : y \in [-1, 1]\} \cup \{(1 - (1/n), 0) : n \in \mathbf{N}\} \cup \{(1, 0)\}$$

with the metric induced by the norm $\|(x, y)\| = |x| + |y|$. Let T be the map $T(0, y) = (0, -y)$, $T(1 - (1/n), 0) = (0, 1 - (1/n))$ and $T(1, 0) = (0, 1)$. We can apply Theorem 1 with condition (i) to T with the choice $p = (0, 0)$, $k_n = 1 - (1/n)$, $n = 1, 2, \ldots$, so that the existence of a fixed point for T is insured, though S is not star-shaped.

Besides, the following corollaries are worth the mention.

COROLLARY 1. *Every continuous $T : S \to S$ where S is compact and star-shaped with reference to $p \in S$ has a fixed point whenever* $\| T((1 - \alpha) p + \alpha T(x)) - T(x) \| \leq \| (1 - \alpha) p + \alpha T(x) - x \|$, $x \in S$, $\alpha \in [0, 1]$.

COROLLARY 2. *Let $T : S \to S$ be such that for a positive number* ϵ, $\| x - y \| \leq \epsilon$ *implies* $\| T(x) - T(y) \| \leq \| x - y \|$. *Then T has a fixed point, whenever S is star-shaped and compact.*

Theorem 2 below is a fixed point theorem for continuous operators with $I - T$ demiclosed, in extension of the result of Dotson ([1], Theorem 2).

THEOREM 2. *Let $T : S \to S$ be continuous where S is weakly compact in the Banach space E. If further T satisfies condition* (i) *or* (ii) *of Theorem* 1, *then T has a fixed point, whenever $I - T$ is demiclosed.*

Proof. S is weakly compact and since the weak topology is Hausdorff, S is strongly closed. S being a closed subset of the Banach space E, it follows that S is complete in the induced metric. As in the proof of Theorem 1, each operator T_n defined by $T_n(x) = (1 - k_n)p + k_nT(x)$ maps S into itself and has a fixed point p_n. The rest of the argument is (an application of the Eberlein–Smulian theorem essentially) exactly as in Dotson's proof of Theorem 2 of [1] and is omitted.

Remarks. As observed by Dotson, it suffices to assume in Theorem 2 that for each $\{x_n\}$ in S weakly converging to x in S and $\{(I - T)(x_n)\}$ converging strongly to 0, $(I - T)(x) = 0$ instead of assuming that $(I - T)$ is demiclosed. Again analogs of the corollaries to Theorem 1 with the additional assumptions that S is a weakly compact subset of a Banach space and that $(I - T)$ is demiclosed are easily deduced.

3. Application to Best Approximation

Theorem 3. *Let E be a normed linear space, V be a finite-dimensional subspace, and $T : E \to E$ having a fixed point f be such that $\|x - y\| \leq d_f(V)$ implies $\|T(x) - T(y)\| \leq \|x - y\|$, where $d_f(V)$ denotes the distance of f from V. If T maps V into itself, then f has a best approximation in V which is another fixed point of T.*

Proof. Since V is finite-dimensional, S_f the set of all best approximations, viz., g lying in V such that $\|g - f\| = \inf_{v\in V}\|v - f\|$, is nonempty. If $g \in S_f$, then $T(g) \in V$, because $S_f \subset V$ and $T(V) \subset V$. $\|T(g) - f\| = \|T(g) - T(f)\| \leq \|g - f\|$ for each $g \in S_f$, in view of $T(f) = f$. Thus T maps S_f into itself. The theorem follows trivially, if the best approximations in V are necessarily unique (i.e., for instance when V is strictly convex).

If S_f is a nontrivial set, then it is well known that S_f is a closed, bounded, and convex subset of V. V being finite-dimensional, it follows that S_f is compact. Applying Corollary 2 of Theorem 1 to the map $T : S_f \to S_f$ we conclude that T has a fixed point in S_f and the proof is complete.

Corollary 1. *If $T : E \to E$ be a nonexpansive operator with a fixed point f and leaving a finite-dimensional subspace V of E invariant, then f has a best approximation in V which is a fixed point of T.*

The following result of Meinardus ([3], Theorem) is easily deduced from Corollary 1.

Corollary 2. *Let $T : B \to B$ be continuous, where B is a compact metric space. If $C[B]$ is the space of all continuous real (or complex) functions on B*

with the supremum norm, let $A : C[B] \to C[B]$ satisfy Lipschitz condition with 1 as a Lipschitz constant. Suppose further that

(i) $A(f(T(x))) = f(x)$,

(ii) $A(h(T(x))) \in V$, *whenever* $h(x) \in V$, *where* V *is a finite-dimensional subspace of* $C[B]$. *Then there is a best approximation* g *of* f *with respect to* V *such that* $A(g(T(x))) = g(x)$.

To deduce Corollary 2 it needs only to be observed that the map $\phi : C[B] \to C[B]$ defined by $\phi(g(x)) = A(g(T(x)))$ satisfies all the hypotheses of Corollary 1 of Theorem 3.

To formulate the subsequent corollaries it is convenient to introduce the concept of a regular function space.

DEFINITION 4. A normed linear space $\langle E, \|\cdot\|\rangle$ of functions on a set X and taking values in a normed linear space $\langle N, \|\cdot\|_1\rangle$ is called A-regular if every operator $T : E \to E$ satisfying the condition $\|T(f_1(x)) - T(f_2(x))\|_1 \leq \|f_1(A(x)) - f_2(A(x))\|_1$, $x \in X, f_1, f_2 \in E$, for some map $A : X \to X$, is non-expansive in the norm $\|\cdot\|$.

A typical example of a space of real functions regular for all mappings is the linear space of all bounded real functions on a set X with the norm

$$\|f\| = a_1 \sup_{x\in X} |f(x)| + a_2[\max_{x\in X} (f(x), 0) + \max_{x\in X} (-f(x), 0)] \qquad (\mathrm{I})$$

where $a_1, a_2 \geq 0$ and not both zero. Similarly B the space of all real-valued bounded measurable functions of a measure space $\langle X, \mathcal{S}, \mu\rangle$ admits of an identity-regular norm given by

$$\|f\| = a_1 \sup_{x\in X} |f(x)| + a_2[\max_{x\in X} (f(x), 0) + \max_{x\in X} (-f(x), 0)] + \sum_{n=1}^{\infty} \frac{a_{n+2}\|f\|_n}{2^{n+2}} \qquad (\mathrm{II})$$

where $\|f\|_n = (\int |f(x)|^n d\mu)^{1/n}$ and a_n is a bounded sequence of real numbers such that a_1 and a_2 are nonzero.

However, not all function-spaces are A-regular for any map A. This is illustrated by the following example.

EXAMPLE 2. Let X be $\{a\}$ and N be the space $\mathbf{R}^2$ with the norm $\|\cdot\|_1$ defined by $\|(x, y)\|_1 = |x| + |y|$. Let E be the space of all maps $f(a)$ of X into $\mathbf{R}^2$ with the norm $\|f(a)\| = \|(f_1(a), f_2(a))\| = \max\{|f_1(a)|, |f_2(a)|\}$. (Thus E is a space of functions taking values in N.) Define $T : E \to E$ as $T(f) = T(f_1(a), f_2(a)) = (f_1(a) + f_2(a), 0)$. For any $f, g \in E$, it is easily seen that $\|T(f) - T(g)\|_1 \leq |f_1(a) - g_1(a)| + |f_2(a) - g_2(a)| = \|f - g\|_1$. However

$\| T(f) - T(g) \| = \text{Max}\{|f_1(a) - f_2(a) - g_1(a) - g_2(a)|, 0\} < \| f - g \| = \text{Max}\{|f_1(a) - g_1(a)|, |f_2(a) - g_2(a)|\}$, for instance, when $g = -f$ where $f = (f_1, f_2)$ and $f_1, f_2 > 0$. Then E is not regular in the sense of the above definition.

COROLLARY 3. *Let $\langle E, \|\cdot\| \rangle$ be an inversion regular normed linear space of functions on $[-1, 1]$ and V, a finite-dimensional subspace of E such that $h(x) \in V$ implies that $h(-x)$ also lies in V. Then every even (odd) function in E has an even (odd) best approximation in V.*

Proof. The operator $T : E \to E$ defined by $T(h(x)) = h(-x)[-h(-x)]$ is nonexpansive as E is inversion-regular. Further T maps V into itself so that in view of Theorem 3, $f(x)$ has an even (odd) best approximation.

It may be seen that the space of all continuous real-valued functions on $[-1, 1]$ with the norm given by (II) is inversion-regular.

The remaining corollaries insure that for certain choices of functions and subspaces 0 is necessarily a best approximation.

COROLLARY 4. *Let $\langle E, \|\cdot\| \rangle$ be a translation-regular nomed linear space of functions on a linear space X and taking values in a normed linear space $\langle N, \|\cdot\|_1 \rangle$. Suppose that*

(i) *f_m, $m = 1, 2, \ldots, n$, are linearly independent functions of E;*
(ii) *$f_m(x + t) = \sum_{k=1}^{n} a_{mk} f_k(x)$ for all $x \in X$ and fixed vector t in X;*
(iii) Det $(a_{mk} - \delta_{mk})$ *is nonzero.*

Then every function f in E periodic with period t has 0 as a best approximation with respect to V, the space spanned by $\{f_m : m = 1, 2, \ldots, n\}$.

Proof. The operator $T : E \to E$ defined by $T(f(x)) = f(x + t)$ maps V into itself in view of (ii) and

$$\| T(f_1(x)) - T(f_2(x)) \|_1 = \| f_1(x + t) - f_2(x + t) \|_1 .$$

Since for fixed t, $x + t$ ranges over X and E is translation regular, T is nonexpansive. Hence by Corollary 1 of Theorem 3, f has a best approximation g in V such that $g(x + t) = g(x)$.

If $g(x) = \sum_{i=1}^{n} b_i f_i(x)$, then, since $g(x) = g(x + t)$,

$$\sum_{i=1}^{n} b_i f_i(x) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_i a_{ik} f_k(x)$$

$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_i a_{ik} \right) f_k(x).$$

So by the linear independence of $\{f_m(x) : m = 1, 2, \ldots, n\}$ we have $\sum_{i=1}^{n} b_i a_{ik} = b_k$ for $k = 1, 2, \ldots, n$. $\sum_{k=1}^{n} (a_{ik} - \delta_{ik}) b_i = 0$ for $i = 1, 2, \ldots, n$. Since Det $(a_{ik} - \delta_{ik}) = 0$, it follows that $b_i = 0$. So 0 is a best approximation of f in V, as required to be proved.

Corollary 5. *Let $\langle E, \|\cdot\|\rangle$ be a strongly regular space of functions defined on a linear space X (and taking values in a normed linear space $\langle N, \|\cdot\|_1\rangle$) in the sense that for any map $h : X \to X$ such that $f_1 \circ h, f_2 \circ h \in E$ for $f_1, f_2 \in E$, $\|f_1 \circ h - f_2 \circ h\| \leq \|f_1 - f_2\|$. Suppose that*

(i) *$g : X \to X$ is such that $f \circ g \in E$ for each $f \in E$;*

(ii) *$f_1, f_2, \ldots, f_n$ are linearly independent elements of E;*

(iii) *$f_m(g(x)) = \sum_{i=1}^{n} a_{mi} f_i(x)$;*

(iv) *Det $(a_{mi} - \delta_{mi}) = 0$.*

Then for any element $f \in E$ such that $f(g(x)) = f(x)$, 0 is a best approximation of f in V, the space spanned by $\{f_i(x) : i = 1, 2, \ldots, n\}$.

Proof. Since $\langle E, \|\cdot\|\rangle$ is strongly regular and T mapping V into itself, defined by $T(f(x)) = f(g(x))$ is nonexpansive, it follows from Corollary 1 that there is a best approximation h such that $h(g(x)) = h(x)$. Because of (iii), $h(x) = 0$ in the same way $g(x) = 0$ in the proof of Corollary 4.

The space of all real-valued continuous functions on a compact topological space X with the supremum norm is strongly regular. However not all function-spaces are strongly regular, as illustrated by the following example.

Example 3. The space of all continuous real functions on $[-1, 1]$ with the L_1-norm is not strongly regular. Let $h : [-1, 1] \to [-1, 1]$ be the map $h(x) = |x|$ and $f_1(x)$ be the function 0 and $f_2(x)$ be defined as

$$\begin{aligned} f_2(x) &= x, && \text{if } x \geq 0, \\ &= x^2, && \text{if } x < 0. \end{aligned}$$

Clearly

$$\begin{aligned} \|f_1 - f_2\| = \int_{-1}^{1} |f_1(t) - f_2(t)|\, dt &= \int_{-1}^{1} |f_2(t)|\, dt \\ &= \tfrac{1}{2} + \tfrac{1}{3} < \|f_1 \circ h - f_2 \circ h\| \\ &= \int_{-1}^{1} |f_2(|t|)|\, dt = 1. \end{aligned}$$

Thus this space is not strongly regular.

4. Examples

Example 4. 0 is a best approximation of $\cos \cos x$ with respect to the subspace spanned by $\cos mx$, $\sin mx$, $m = 1, 3, 5, \dots, 2n + 1$, for any non-negative integral n in the space of all bounded real-valued functions of a real variable under any translation regular norm, as seen from Corollary 4 by taking $t = \pi$. This in turn implies that 0 is still a best approximation in the space spanned by $\{\cos (2m + 1) x, \sin (2m + 1) x : m = 0, 1, 2, \dots\}$.

Example 5. Let B be the space of all bounded real-valued functions of a real variable with the supremum norm. Let $r_1, r_2, \dots, r_n$ be n distinct real numbers and $f_0(x)$ be a function of a real variable with $\inf_{x \in \mathbf{R}} f_0(x) > r_i$ for $i = 1, 2, \dots, n$. Taking $g(x) = f_0(x)$, $f_i =$ the characteristic function of $\{r_i\}$ and V the linear span of these functions $\{f_i : i = 1, 2, \dots, n\}$, the conditions of Corollary 5 are readily verified. (For, $f(g(x)) = 0$, $f \in V$, and (iv) of Corollary 5 is obvious as each $a_{mi} = 0$.) Consequently for any function f in $B - V$ and satisfying $f(g_0(x)) = f(x)$, 0 is a best approximation in V by Corollary 5.

For instance, one may choose $r_i = -i$, $i \in \mathbf{N}$, and $i = 1, 2, \dots, n$ with $f_0(x) = |x|$. In this case, it follows that 0 is a best approximation in V for any bounded even function in $B - V$.

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